Spiral Tilings with C-curves
Using Combinatorics to Augment Tradition

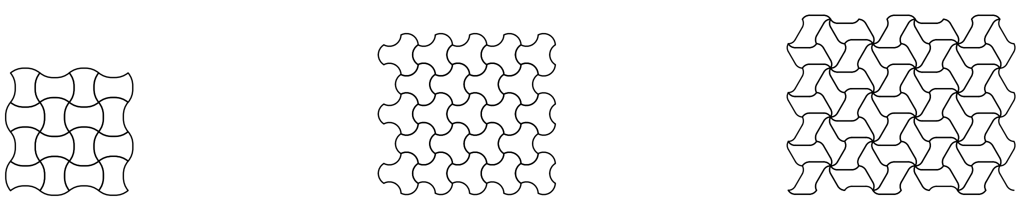
Chris K. Palmer
319 North Albany Avenue
Chicago, Illinois, 60612
chris@shadowfolds.com
www.shadowfolds.com

Abstract

Spiral tilings used by artisans through the ages using C-curves are identified. The complete combinatoric set of these tiles is determined and tilings that can be made with them are examined.

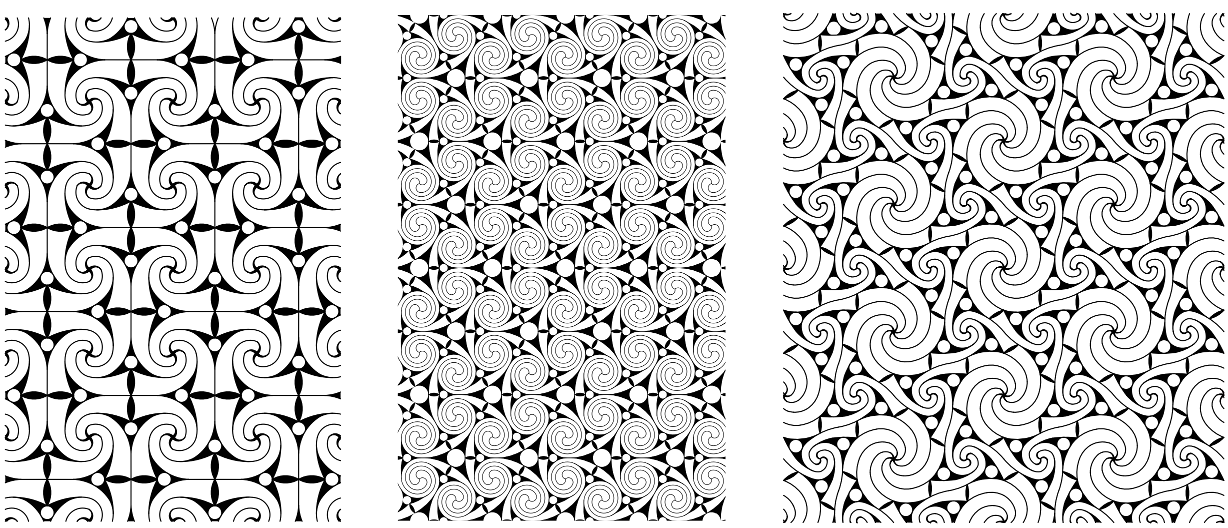
1. Introduction

Throughout the history of ornament artisans from many cultures have composed patterns utilizing spirals. Many of the compositions made by these artisans can be understood as the substitutions of C-curves or S-curves or a combination of C-curves and S-curves for the straight lines of a monohedral or polyhedral tiling [1]. The subject of this paper is the subset of these tilings composed with C-curves. Those known and used by traditional craftsmen are shown below in Figures 1 and 2 [2]. Artisans can apply and have applied an enormous variety of styles using these three tilings. An example of this variety can be seen by observing the expression of the same tiling, (4.4.4.4), in two different styles, a simple form in Figure 1a and a Celtic style in Figure 2a. These underlying tilings are part of a set of tilings with regular vertices commonly known as Laves tilings [3]. This paper identifies the complete set of tiles and examines some properties of the tilings that can be made with them. This greatly augments the number of underlying tilings available for use by artisans from three to an infinite number of tilings [4].



(a) (b) (c)

Figure 1



(a) (b) (c)

Figure 2

2. The Problem

In general terms, our problem is:

What tilings of the plane allow for C-curves to be substituted for the edges of the tiles?

We will restrict the problem by imposing the following conditions:

i) Spirals must be produced at all the vertices.
ii) The angles surrounding any vertex of the tiling must all be equal. [3]

As we shall see these tilings are very restricted, but there are nevertheless such a large number of different kinds that we can expect to be able to classify them only in some weak sense. As will be described a third condition results:

iii) Each tile must have an even number of sides.

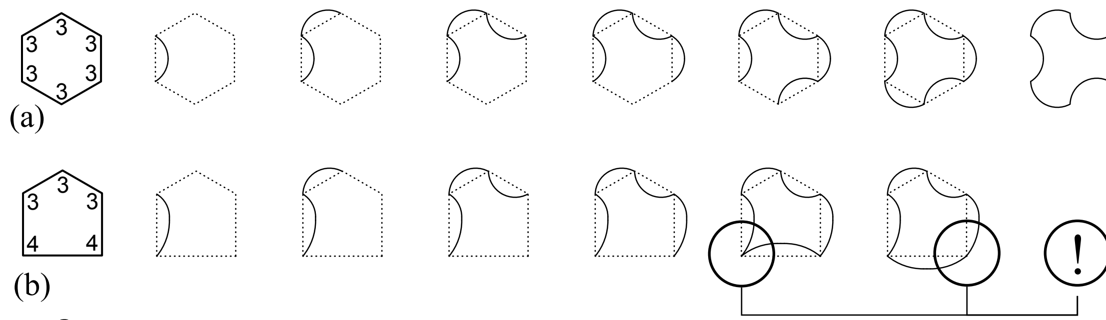


Figure 3

Condition ii) shows that the angles at a vertex that lies on *n* edges must be equal to 360°/*n*. Condition iii) holds because the spiral property i) entails that for any two C-curves that will be substituted for adjacent edges of a tile one will lie inside the polygon and one outside. This alternation can only be consistently continued around the whole tile if that tile has an even number of sides as shown in Figure 4a. The tile in Figure 4b satisfies condition ii) but not condition iii) causing contradiction of condtion i). Conversely, however, if we are given any tiling satisfying conditions ii) and iii), then after assigning a clockwise or counterclockwise orientation arbitrarily to any one vertex, we can consistently assign such orientations to the other vertices by alternation. This shows that in general there are two ways of orienting the spirals for a given tiling.

3. The Possible Types of Tile

We shall say that a vertex has a type *n* when it belongs to just *n* tiles, which will all have angles of 360°/*n* there. An edge whose two vertices have types *a* and *b* will be said to have type (*a*.*b*), and a tile will be said to have type (*a*.*b*.*c*. , ... *k*) if its vertices, reading cyclically (in either direction) have types *a*,*b*,*c*,...,*k*.

Theorem 1. *The only possible types of polyhedral or monohedral tiles that cover the plane and allow C-curves to be substituted for the lines and satisfy conditions i), ii) and iii) are:*

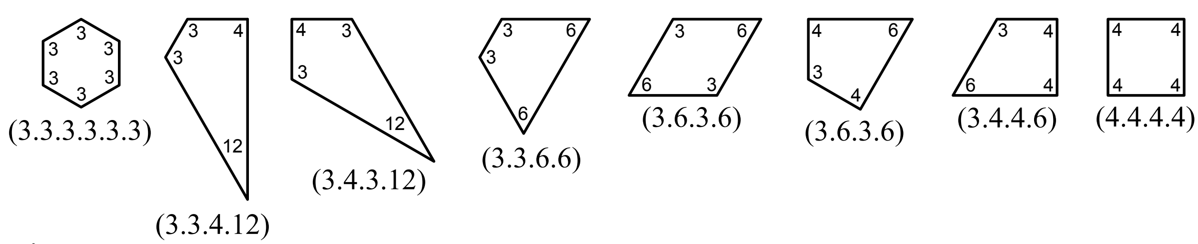


Figure 4

Proof. Since the interior angle of a tile at an *n*-fold vertex is 360°/*n*, the exterior angle is 180°– 360°/*n*. Now for:

$n =$	3	4	5	6	7	8	9	10	11	12	>12
$360/n =$	120°	90°	72°	60°	51.4°	45°	40°	36°	32.7°	30°	<30°
$180 - 360/n =$	60°	90°	108°	120°	128.5°	135°	140°	144°	147.2°	150°	>150°

Table 1

The sum of the exterior angles of a polygon is always 360°! Since Table 1 shows that the smallest value for an exterior angle in these tiles is 60°, there can be at most 6 angles per tile, and if there are 6, that tile must be of type (3.3.3.3.3.3).

Otherwise, since the tile must have an even number of sides, it must in fact be a quadrilateral, with an average exterior angle of 90°, and a smallest one of either 60° or 90°. If there are (at least) two of 60°, the sum of the remaining two is 240°, and from Table 1 we have only two possibilities:

90° + 150°, corresponding to types (3.3.4.12) and (3.4.3.12)

and

120° + 120°, corresponding to types (3.3.6.6) and (3.6.3.6).

If there is precisely one such 60° exterior angle, the sum of the other three must be 300°, which forces the least to be 90°, forcing another 90° angle and leaving precisely 120° for the last one, corresponding to:

(3.4.4.6) and (3.4.6.4).

Finally, if there is no 60° angle, all four angles must be 90°, since this is now both the smallest possible value and also their average value, and we have the only remaing type:

(4.4.4.4).

and this completes the proof. □

We remark that the angles of these tiles do not suffice to determine their shapes: for instance a tile of type (4.4.4.4) might be any shape of rectangle. However, we shall regard a tiling as well-enough determined when we have specified all the angles of its tiles. There is then a canonical tiling with these angles, obtained by using tiles with the particular dimensions shown in Figure 4. From now on we shall always draw only these canonical tilings.

4. Tilings With a Type 12 Vertex.

Theorem 2. *No tiling involves a tile of type (3.4.3.12), and there is combinatorially only one tiling that involves one of type (3.3.4.12).*

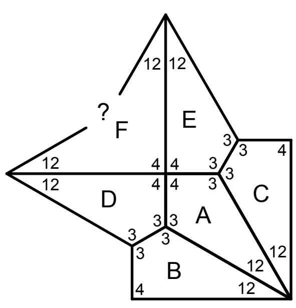


Figure 5

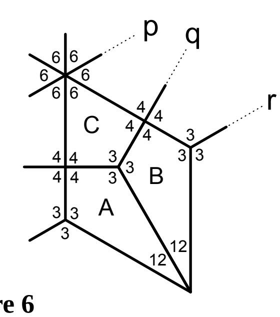


Figure 6

Proof. We examine the type 12 vertex of a tile A of type (3.4.3.12) shown in Figure 5. If both the adjacent tiles B and C at this vertex have tye (3.3.4.12) then the tiles D and E of Figure 3 are forced to have types (3.3.4.12), and the tile F between them has an impossible type (12.4.12.?). So we can suppose that B has type (3.4.3.12) as in Figure 6, and this forces the tile C of that Figure 6 to have type (4.3.4.6). The numbers p and q of that Figure must be 3 and 4 in some order. But if q = 3, then r = 12, making tile F to have the impossible shape (12.3.12,?). So we must have p = 3, q = 4, and so r = 6, and again F has an impossible type, (6.3.12.?).

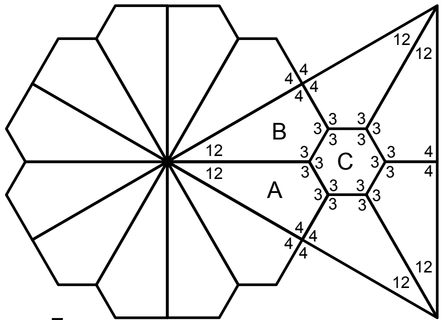


Figure 7

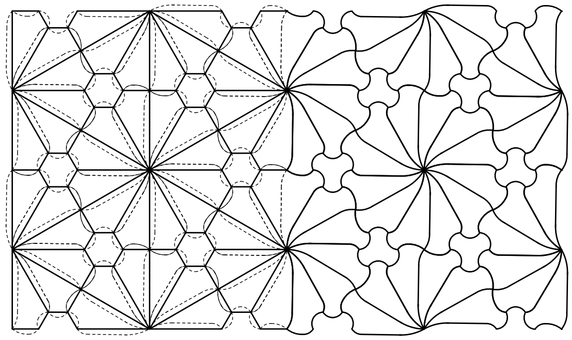


Figure 8

Now that we have dismissed type (3.4.3.12), we see that all the tiles with any type 12 vertex of type (3.3.4.12) must have that type, forming a rosette as in Figure 7 and the tile C of that Figure has a type (3.3.3....) which can only be (3.3.3.3.3.3). This in turn forces tiles D and E to have type (3.3.4.12), yielding two more rosettes around vertices Y and Z. Now it is clear that the tiling continues uniquely as in Figure 8 thus concluding the proof. □

5. Tilings with Edges of Type (3.3) or (6.6)

Theorem 3. *If a tiling has an edge of type (3.3) or (6.6), it must be one of the tilings H, H1, H2, .., H∞, or H∞* of Figures 9-12.*

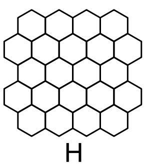


Figure 9

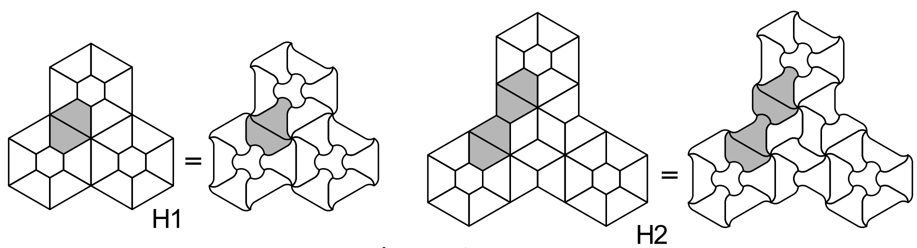


Figure 10

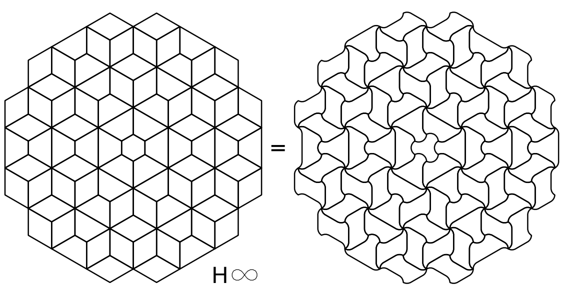


Figure 11

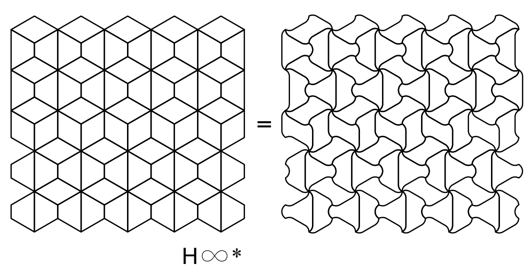


Figure 12

Proof. The only types that contain two adjacent 3's are (3.3.3.3.3.3) and (3.3.6.6). The latter is also the only type containing two adjacent 6's. So it follows that if one edge of a tile is of type (3.3), then the opposite edge is either (3.3) or (6.6), while the edge opposite to one of type (6.6) is necessarily of type (3.3).

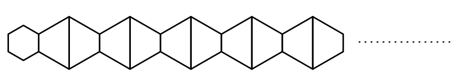


Figure 13

The tiles containing all these opposite pairs of edges therefore form "accordion-rows" like that of Figure 13, each such row consisting of an infinity of hexagons each of which is either a single tile of type (3.3.3.3.3.3) or is split into two tiles of type (3.3.6.6).

We suppose first that some accordion-row does contain two unsplit hexagons, and select one for which these hexagons (A1 and A2) are separated by the smallest possible number of tiles (B) that happens in that

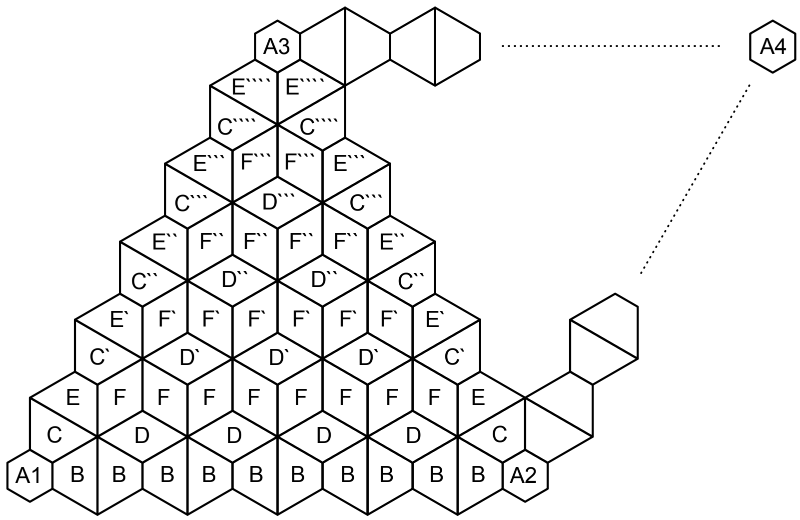


Figure 14

tiling (Figure 14). Then the two tiles marked C in the Figure have a type (3.3.6. ?) that can only be (3.3.6.6), and we can also determine the type of the tiles marked D, since (6.3.6. ?) can only be (6.3.6.3).

We can proceed upwards in this way, forcing all the types of tiles (and so the shape of the Figure) until we conclude triumphantly that there must be a third hexagonal tile at A3. In the third row, for example, we can see that the end tiles E have a type (6.6.3.?) = (6.6.3.3), and this forces type of F1 to be (3.6.3.?) = (3.6.3.6), which in turn forces F2 to to have type (6.3.6.?) = (6.3.6.3), and so on. The argument then repeats, with a row of tiles C' and D' behaving like C and D followed by one of tiles E' and F' behaving like E and the Fi, and so on, until just below the summit we have a "row" of just two tiles C''''... surmounted by one of just two tiles E''''... . The tile A3 at the summit has type (3.3.3. ?...) which can only be a hexagon (3.3.3.3.3.3).

We have found an "equilateral triangle" of tiles whose "vertices" are A1, A2, and A3. But we could repeat the argument to show that abutting this "triangle" along its "edge" A2 A3 there is another one, with vertices A2 , A3, A4. In the case illustrated in Figure 14, the centers of any two nearest unsplit hexagons are separated by 6 "units", and so we call this the tiling H6. But the separation could be any positive integer.

The argument also handles the case when an accordion-row contains only one unsplit hexagon which is natural to call H∞ (Figure 11), and that in which it has no unsplit hexagon, H∞* (Figure 12). This concludes the proof. □

6. The Tiling With No Vertex of Type 4

At any time from now on, the tilings we have already found will be called the old tilings. Any new tiling can have tiles of only the types:

(3.6.3.6), (3.4.4.6), (3.4.6.4), (4.4.4.4).

Theorem 4. *The only new tiling with no vertex of type 4 is the rhombic tiling R of Figure 15.*

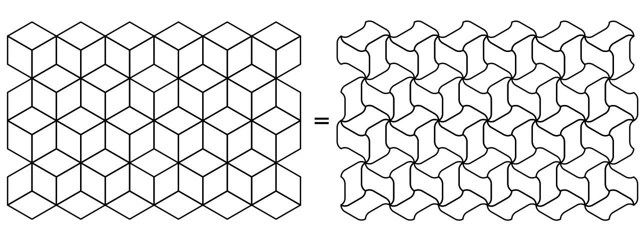


Figure 15

Proof. Plainly all its tiles must have type (3.6.3.6), and a moment's thought suffices to show that these can only abut as in Figure 15. □

7. Tilings With a Line of Type (...,4.4.4.,...)

We say that a straight line has type (...,4.4.4.,...) if it consists entirely of type (4.4) edges of the tiling.

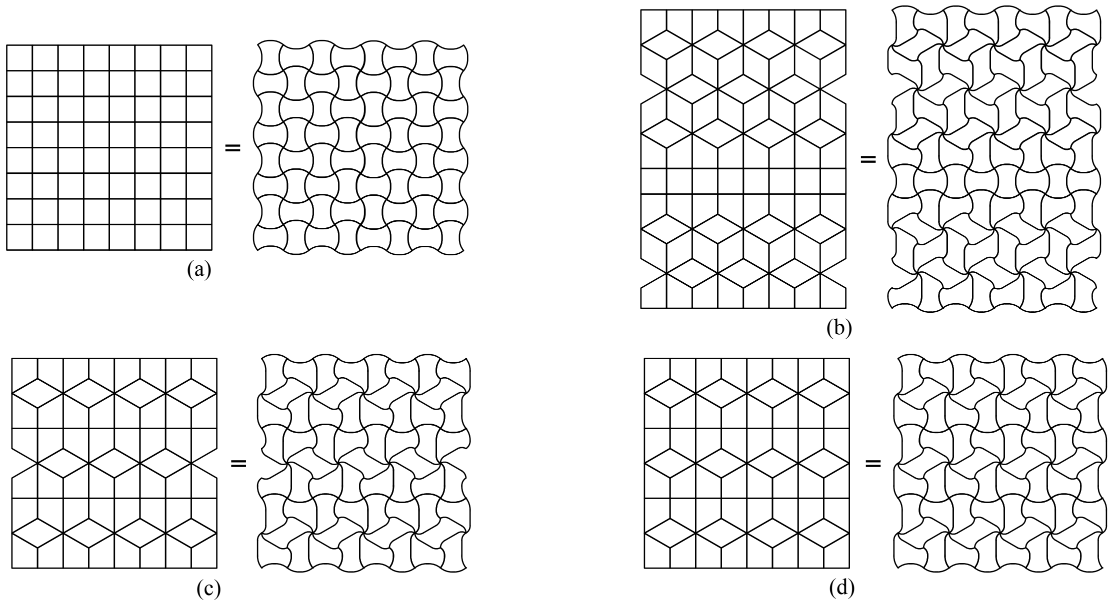


Figure 16 (*Some examples of many possible arrangements of "parallel tilings"*)

Theorem 5. *The only tilings containing such a line are the "parallel tilings" illustrated in Figure 16.*

Proof. We consider the tiles that abut the given line on one side. Any chosen one of these has a type (4.4.?.?) which can only be (4.4.4.4) as in Figure 17 or (4.4.3.6) as in Figure 18. In the first case the tiles neighboring the given one have shapes (4.4.4.?) which makes them also (4.4.4.4), and inductively we see that in fact all the tiles on that side have that type, and are bounded above by another of type (...,4.4.4.,...).

Figure 17

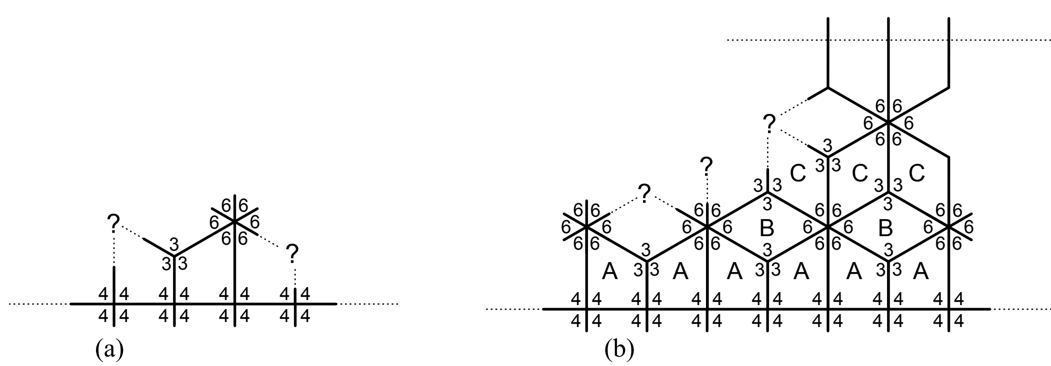


Figure 18

In Figure 18a the neighboring types are (?.4.4.3) or (6.4.4.?), and so are necessarily (6.4.4.3), and we see again that all the tiles (A) on that side of the line have this type. But this time they are bounded above by a zig-zag line whose vertices are alternately of types 6 and 3. The tiles (B) that abut them from above have types (6.3.6.?) = (6.3.6.3), and we obtain Figure 18b.

The tiles (C) above these have types (?.6.3.?) which may be either (4.6.3.4) or (3.6.3.6), and it is again easy to see that in fact all of them must have the same one of these types. If this is (4.6.3.4) they are bounded above by another line of type (...,4.4.4.,...) (hinted at by the dotted line in Figure 18b). If not, we have another row (D) of horizontal rhombs (6.3.6.3), and so on. □

8. Patches and the Parity Rule

From now on, we shall separate the edges of our tilings into two classes, indicated by thick and thin lines. We make this decision arbitrarily for one edge, and then continue by using the rule that any pair of adjacent edges of a tile are to be the same thickness if they abut at a type 3 or 6 vertex, but of different thickness if they abut a type 4 vertex. This rule is consistent around any tile because all possible types have an even number of type 4 vertices, and around any vertex because we can make its edges all have the same thickness if it has type 3 or 6, or alternating thicknesses if it has type 4.

The thick edges now divide the tiling into regions which we shall call patches, consisting of various numbers of tiles. The edges of a patch will consist of chains of edges of the underlying tiling, with possibly some internal verticies which must all be of type 4 and end vertices which must be of type 3 or 6. Plainly any vertex in the interior of a patch must have type 3 or 6, and two adjacent such vertices must be one of each type (in a new tiling). An example is shown in Figure 19.

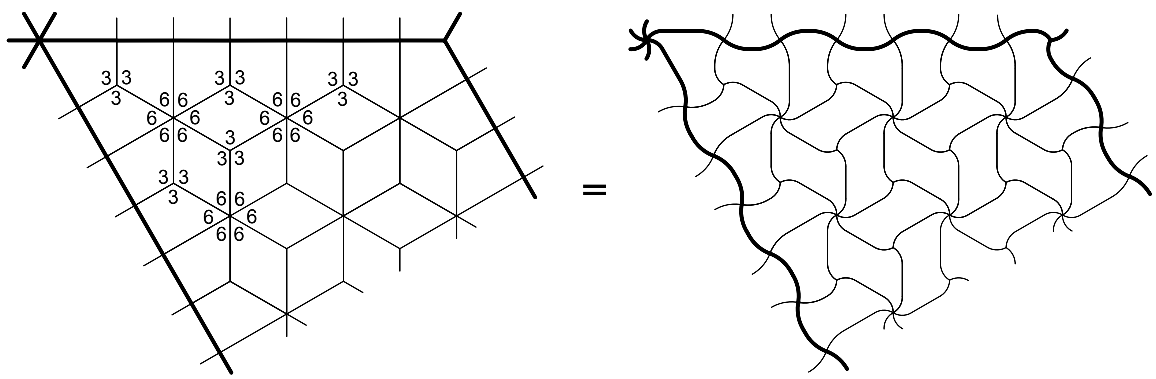


Figure 19

We shall now prove what we call "The Parity Rule":

Theorem 6. *The two ends of an edge of a patch that has any internal vertices are of the same type (3 or 6) if that edge has an odd number of internal vertices; of opposite types (one 3, one 6) if it involves an even number.*

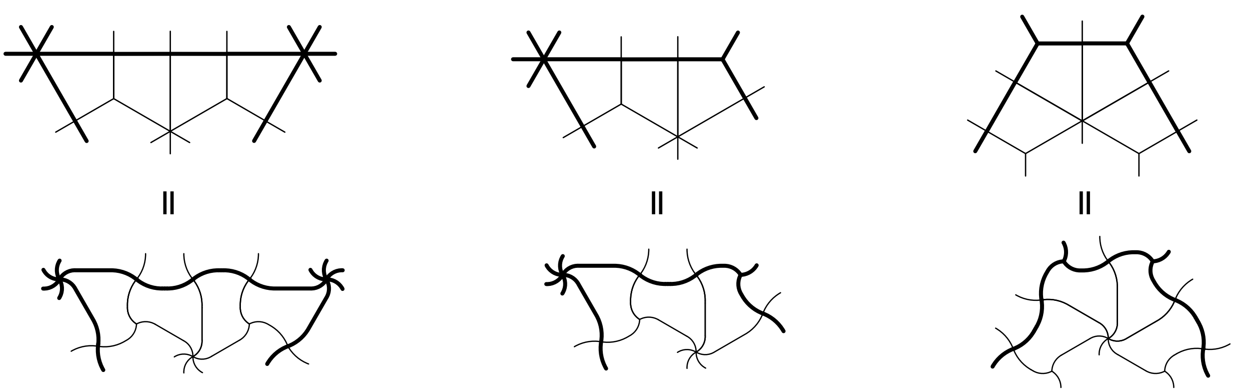


Figure 20

Proof. As in the proof of Theorem 5, the tiles that abut this edge inside the patch must all have type (6.4.4.3) except for the two at the ends. Figure 20 shows three possibilities: we see that these tiles are bounded on the side away from the edge of the patch by a zigzag line whose ends have type 4, while the others have alternately types 3 and 6. Since the only possible type for the two end tiles is (4.3.4.6) the ends of the edge must have the types opposite to those of the outermost internal vertices of this zigzag, which establishes the theorem. □

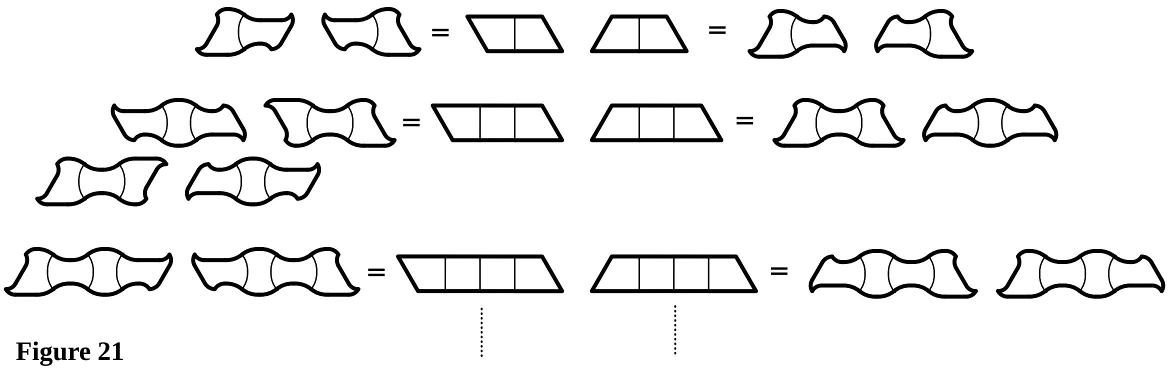


Figure 21

There can exist patches that violate the parity rule, but they may contain no internal vertices. It is easy to see that any patch containing no internal vertices is of one of the shapes shown in Figure 21. (Not all of these violate the parity rule.)

Theorem 7. *Any tiling with a patch that violates the parity rule is one of the Parallel Tilings found in Theorem 5.*

Proof. If one edge of a patch violates the Parity Rule, then we see from Figure 21 that so does the opposite edge. The patches adjacent to the given one along these edges must also be in Figure 21, since they equally violate the rule. We can continue in this way to obtain an infinite "concertina" of such patches, which we see contains some lines of type (....,4.4.4.,....), and so must be covered by Theorem 5. So no new Tiling can contain any edge that violates the Parity Rule. This concludes the proof. □

9. The Difference Rule, and the Shapes of Patches

Since each vertex angle of a patch is either 60° or 120°, a patch can be at most hexagonal. We shall state and prove the Difference Rule first for hexagonal patches, since this is in a way the most general case.

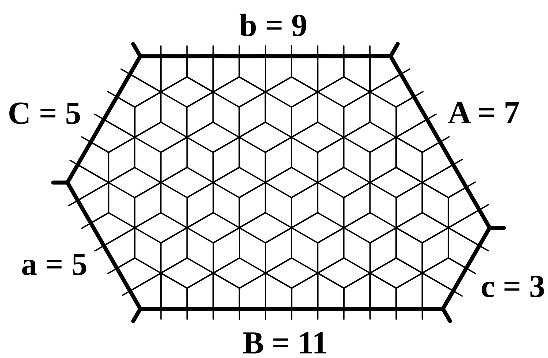


Figure 22

Theorem 8. *For a hexagonal patch, the numbers A,b,C,a,B,c of lines that cross the successive edges of the patch (see Figure 22) obey the "Difference Rule":*

$$A - a = B - b = C - c.$$

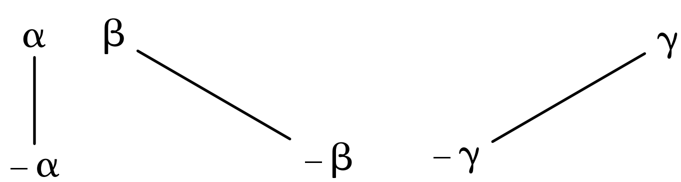


Figure 23

Proof. We introduce three "scores" α , β , γ, that are attached to the ends of the thin edges of tiles according to the direction in which they point, as in Figure 23. Since the two ends of any edge cancel, the total scores of ALL the edge-ends in our patch must all be zero.

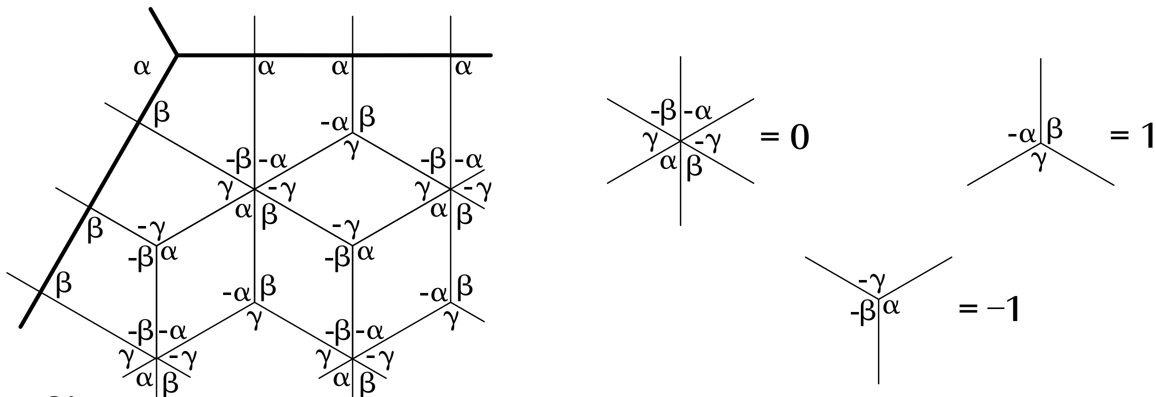


Figure 24

Now as seen in Figure 24 the three edge-ends at a type 3 vertex have either:

$$\alpha = \beta = \gamma = 1 \quad \text{or} \quad \alpha = \beta = \gamma = -1\,,$$

while the totals for the six edge-ends at a type 6 vertex are

$$\alpha = \beta = \gamma = 0.$$

It follows that if we sum over all edge-ends at all internal vertices of a patch, the three scores satisfy

$$\alpha = \beta = \gamma\,,$$

and therefore the same must be true for the scores of the edge-ends at boundary vertices of the patch. But for these the scores are

$$\alpha = A - a,\quad \beta = B - b,\quad \gamma = C - c,$$

proving the rule.

The possible shapes of patch are shown in Figure 25. The second and third rows of this figure show the smallest patches possible which do not contain the rhombic tiling R. The finite ones can be regarded as hexagons in which some edges have length zero (and so contribute zero to the appropriate score), and so the Difference Rule and its proof continues to hold for them.

The infinite patches shown in Figure 26 can also be regarded as hexagons in which some edges have infinite length, while others have receded to infinity. The proof of the Difference Rule fails, because the sums involved become infinite; and in fact those of the numbers A,b,C,a,B,c that survive are only restricted by the Parity Rule. This concludes the proof. □

10. All Remaining Tilings

All the tilings not already discussed are obtained from tessellations of the plane by pieces of the shapes indicated in Figure 25 in the following way. We attach to each finite edge a number, which is the number of thin edges which that edge is to cross. These numbers must satisfy the Parity Rule, and for finite patches also the Difference Rule. If they do so, they determine a unique tiling, in which each patch is filled by a portion of the rhombic tiling R. These patches may be combined with the patches shown in Figure 21.

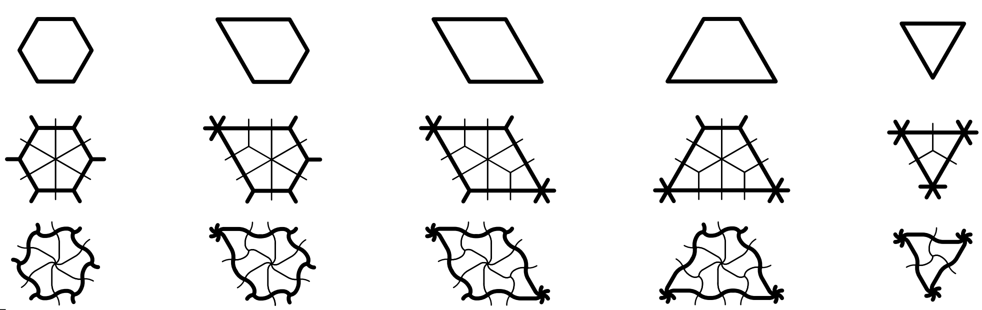


Figure 25

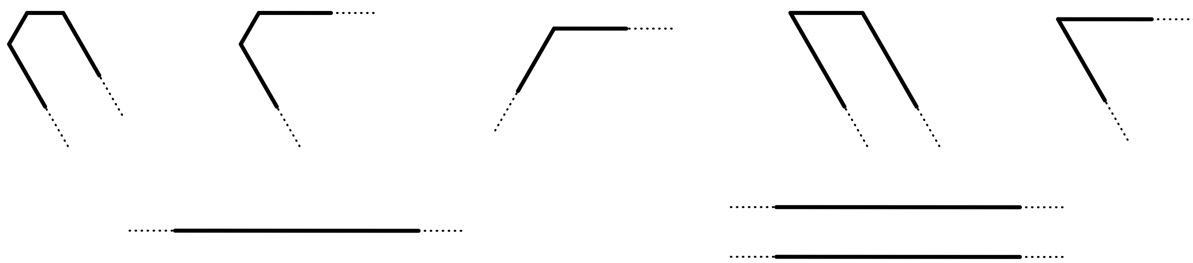
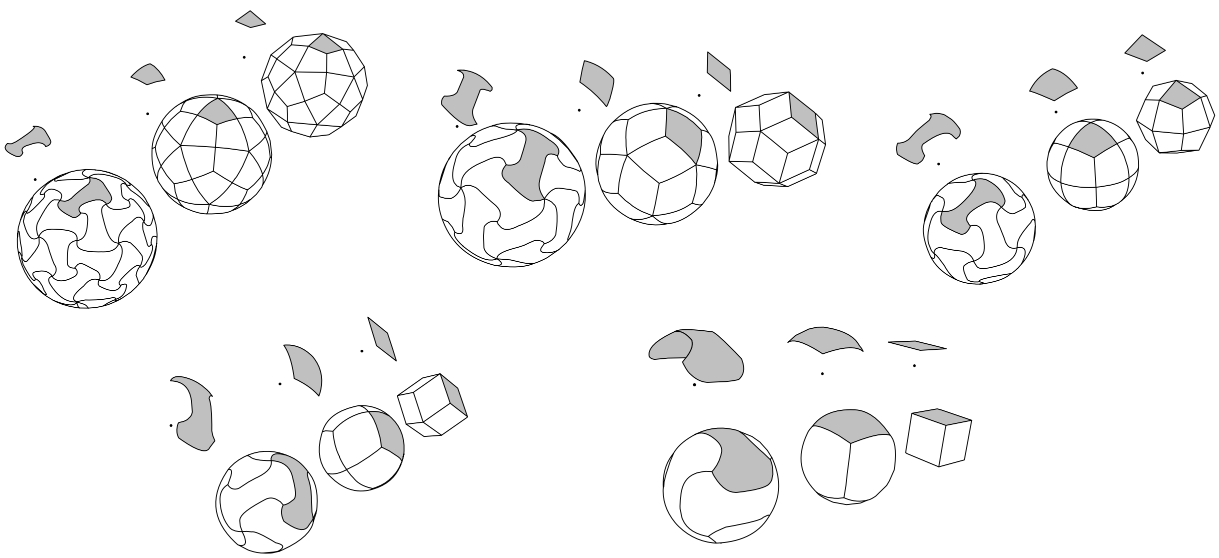


Figure 26

One example of a composition using a traditional Celtic style to express one of the new underlying tilings presented in this paper can be seen at the front of the Bridges Section. Below are the five spherical tilings with regular vertices that allow C-curves.



Acknowledgment

I thank Professor John Conway of the University of Princeton for spending hours of his time to help put my arguments into mathematical language and dedicate this paper to him.

References

[1] The complete set of tiles that includes S-curves, C-curves and combinations of S-curves and C-curves, a study of those compositions used by traditional artisans and many that augment tradition utilizing them is presented in *Scurls and Whirl Spools* by Chris K. Palmer, VisMath Volume 10.
[2] Some interesting examples of traditional compositions that use tilings with irregular underlying tilings not addressed by this paper can be seen in *Celtic Art the methods of Construction*, George Bain, Dover ISBN 0-486-22923-8, page 66 plates 11-12. Pages 64-65 Plates 7-10 contain examples of frieze patterns and rosettes utilizings spirals also not addressed in this paper.
[3] After the famous crytallographer Fritz Laves. See *Tilings and Patterns*, Grunbaum and Shepard, ISBN 0-7167-1193-1, pages 95-98
[4] Recursive methods developed by the author from 1997 to the present show that within the restrictions imposed by the rules described in this paper an infinite variety of spiral tilings can be composed. These will be presented in forthcoming papers and as part of a doctoral thesis. The principles presented in this paper apply for tilings on spheres: http://www.shadowfolds.com/websphere/wspools.htm. Hyperbolic tilings can also be composed with these tiles and the addition of other even sided tiles.